

MAPPING SPACES AND POSTNIKOV INVARIANTS

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ABSTRACT. If $q : Y \longrightarrow B$ is a fibration and Z is a space, then the free range mapping space $Y!Z$ has a collection of partial maps from Y to Z as underline space, i.e. those such maps whose domains are individual fibre of q .

It is shown in [4] that these spaces have applications to several topics in homotopy theory. These such results are given in complete detail, concerning identification, cofibrations and sectioned fibrations. The necessary topological foundations for two none complicated applications, i.e. to the cohomology of fibrations and the classification of Moore-Postnikov systems, are given, and the applications themselves outlined.

The usual argument is in the context of the usually category of all topological spaces, and this necessarily introduces some new problems. Whenever we work with exponential laws for mapping spaces, in that category, we will usually find that we are forced to assume that some of the spaces are locally compact and Hausdorff, which detracts considerable from the generality of the results obtained.

In this paper we develop the corresponding theory in the category of compactly generated or k -spaces, which is free of the inconvenient assumptions referred to above. In particular, we obtain the k -space version of the applications to identifications, cofibrations and sectioned fibrations, and establish improved foundations for the k -versions of the other two applications, i.e. the cohomology of fibrations and a classification theory for Moore-Postnikov factorizations.

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1. COMPACTLY GENERATED SPACES k – spaces

1.1. Universal Property for k – spaces. Let X be a topological space. We define kX to be the space X retopologized with the final topology (see [23]) relative to all incoming maps from compact Hausdorff spaces. Thus if

$$g : C \longrightarrow X$$

is continuous, then

$$g : C \longrightarrow kX$$

is continuous, in fact kX has the finest topology for which all maps

$$g : C \longrightarrow X$$

are continuous. If $kX = X$ then X will be said to be a *compactly generated space* or k – space. We will refer to kX as the k – ification of X . For more detail, concerning k – spaces see [5].

Theorem 1. Universal Property of the space kX Let X , and Y be spaces and $h : X \longrightarrow Y$ be a function. Then the composite functions $h \circ g : C \longrightarrow X \longrightarrow Y$ are continuous, for all

$$g : C \longrightarrow X$$

with C compact Hausdorff, if and only if

$$h : kX \longrightarrow Y$$

is continuous.

Proof. If

$$h : kX \longrightarrow Y,$$

and

$$g : C \longrightarrow X$$

are continuous, then

$$g : C \longrightarrow kX$$

is continuous, and $g \circ h$ is continuous. Conversely, let $h \circ g$ be continuous, for all maps

$$g : C \longrightarrow X,$$

where C is compact Hausdorff, we wish to prove that $h : kX \longrightarrow Y$ is continuous. Let U be open in Y . Then

$$(h \circ g)^{-1}(U) = g^{-1} \circ h^{-1}(U)$$

is open implies that $h^{-1}(U)$ is open in kX , since kX has the final topology with respect to all maps

$$g : C \longrightarrow kX.$$

□

Proposition 1. *The identity function*

$$1 : kX \longrightarrow X$$

is continuous, for all spaces X .

Proof. For any map

$$f : C \longrightarrow kX,$$

$1 \circ f$ is continuous, so $1 : kX \longrightarrow X$ is continuous by the Universal Property.

□

Proposition 2. *If*

$$f : X \longrightarrow Y$$

is continuous, and X and Y are spaces, then

$$kf : kX \longrightarrow kY$$

where $kf(x) = f(x)$ is continuous.

Proof. Let C be compact Hausdorff space, and

$$g : C \longrightarrow X$$

be continuous. Then

$$f \circ g : C \longrightarrow Y$$

is continuous. Hence

$$kf \circ g : C \longrightarrow kX \longrightarrow kY$$

is continuous, for any incoming map

$$g : C \longrightarrow X.$$

It follows that

$$f \circ g : C \longrightarrow kY$$

is continuous, and that kf is continuous by the Universal Property.

□

Proposition 3. *If X is k -space, and Y is any space, then $f : X \longrightarrow Y$ is continuous if and only if*

$$f' : X \longrightarrow kY$$

is continuous, where $f'(x) = f(x)$ for all $x \in X$.

Proof. Let f be continuous. Then

$$f : kX = X \longrightarrow kY$$

is continuous by the previous proposition.

Conversely, let

$$f' : X \longrightarrow kY$$

be continuous. Then

$$f = 1 \circ f' : X \longrightarrow Y$$

is continuous, where

$$1 : kY \longrightarrow Y$$

is the continuous identity function (see Proposition 2). \square

Proposition 4. *If C is a compact Hausdorff space, then a function $g : C \longrightarrow X$ is continuous if and only if*

$$g : C \longrightarrow kX$$

is continuous.

Proof. The *only if* part follows from the definition of kX , as was explained on page 1.

Conversely, let

$$g : C \longrightarrow kX$$

be continuous. Then

$$1 : kX \longrightarrow X$$

is continuous, and so

$$g : C \longrightarrow X$$

is continuous. \square

Proposition 5. *If C is compact Hausdorff space, then C is k – space.*

Proof. From Proposition 1,

$$1 : kC \longrightarrow C$$

is continuous, and the identity function

$$g : C \longrightarrow kC$$

is continuous by Proposition 4. Then $kC = C$, and C is a k – space. \square

Proposition 6. *If X is any space, then $kX = k(kX)$.*

Proof. The proof lies in the observation that kX , and $k(kX)$ have the final topology relative to all maps

$$g : C \longrightarrow X$$

and all maps

$$g : C \longrightarrow kX$$

respectively, and by Proposition 4 these are the same maps in each case. \square

Corollary 1. *For any space X , kX is a k – space.*

Proof. From the previous proposition. \square

1.2. CW-Complexes.

Proposition 7. *If Y has the final topology with respect to a family of functions*

$$\{f_j : X_j \longrightarrow Y\}_{j \in J},$$

where all X_j are k -spaces, then Y is a k -space.

Proof. Let

$$g : C \longrightarrow X_j$$

be continuous, for all $j \in J$, with C compact Hausdorff. Then

$$f_j \circ g : C \longrightarrow Y$$

is continuous, and if U is open in kY , then

$$(f_j \circ g)^{-1}(U)$$

is open in C , by the definition of final topology. Thus

$$g^{-1}(f_j^{-1}(U))$$

is open in C . Hence $f_j^{-1}(U)$ is open in $kX_j = X_j$ for all $j \in J$ by the definition of final topology. Then U is open in Y , again by the definition of final topology, and so $Y = kY$ as we required. \square

Corollary 2. *If*

$$f : X \longrightarrow Y$$

is an identification, and X is a k -space, then Y is a k -space.

Corollary 3. *If $\{X_j\}_{j \in J}$ is a family of k -spaces, then the disjoint topological sum*

$$\coprod_{j \in J} X_j$$

is a k -space.

Corollary 4. *Every CW-complex is a k -space.*

Proof. If $\{D_j\}_{j \in J}$ are the cells of a CW-complex K , and the inclusion

$$D_j \hookrightarrow K$$

is denoted by i_j , then K has the final topology relative to the family

$$\{i_j : D_j \longrightarrow K\}_{j \in J}.$$

We wish to prove that $K = kK$. We know that the identity

$$kK \longrightarrow K$$

is continuous, so we just have to prove the continuity of the identity $1 : K \longrightarrow kK$. Then $1 \circ i$ is a map from a compact Hausdorff space into K and so, by Proposition 4, is continuous. It follows by Theorem 1 that

$$1 : K \longrightarrow kK$$

is continuous, and so $K = kK$, and K is a k -space. \square

1.3. Initial Topologies on X .

Remark 1. Let X carry the initial topology (see [23]), relative to the family of functions

$$\{g_j : X \longrightarrow X_j\}_{j \in J}.$$

If the spaces X_j are k -spaces, it does not necessarily follow that X is a k -space. The product space $Y \times Z$ carries the initial topology relative to the projections

$$p_1 : Y \times Z \longrightarrow Y,$$

and

$$p_2 : Y \times Z \longrightarrow Z,$$

yet there is well known examples in [6] and [8], where Y and Z are CW-complexes, yet $Y \times Z$ is not a k -space. However, the following result tells us that the k -ification of the usual sense initial topology is the appropriate model for a k -space initial topology on X .

Theorem 2. The Universal Property for k -spaces Initial Topologies on X Let $\{X_j\}_{j \in J}$ be a family of k -spaces, and X carry the initial topology in the usual sense relative to a collection of functions

$$\{g_j : X \longrightarrow X_j\}_{j \in J}.$$

Then kX is the initial topology of X in the k -sense, as can be seen from the following Universal Property.

(a) The functions

$$g_j : kX \longrightarrow X_j$$

are continuous, and

(b) If W is a k -space and

$$h : W \longrightarrow X$$

is a function, then

$$h : W \longrightarrow kX$$

is continuous if and only if the composites

$$g_j \circ h : W \longrightarrow X_j$$

are continuous, for all $j \in J$.

Proof. (a) follows from Proposition 2, (b) from Proposition 2, and the Universal Property of initial topologies in the usual sense. \square

Remark 2. If X and Y are sets, then a function

$$\alpha : W \longrightarrow X \times Y$$

is of the form $\langle \alpha_1, \alpha_2 \rangle$, where

$$\alpha_1 : W \longrightarrow X,$$

and

$$\alpha_2 : W \longrightarrow Y.$$

Thus $\alpha(w) = \langle \alpha_1, \alpha_2 \rangle (w) = (\alpha_1(w), \alpha_2(w))$, for all $w \in W$.

If W , X and Z are spaces, then the Universal Property of products spaces asserts that α is continuous if and only if α_1 , and α_2 are continuous.

We define $X \times_k Y = k(X \times Y)$. For k -spaces X and Y , it follows from Theorem 2 that $X \times_k Y$ is the product of X and Y in the k -sense.

Remark 3. Given maps

$$p : X \longrightarrow B,$$

and

$$q : Y \longrightarrow B,$$

then we will define the pullback space or fibred product space of X and Y , to be the subspace of $X \times Y$ with underlying set

$$X \sqcap Y = \{(x, y) | p(x) = q(y)\}.$$

In this situation

$$p^*q : X \sqcap Y \longrightarrow X,$$

and

$$q^*p : X \sqcap Y \longrightarrow Y$$

will be denote the corresponding induced projections. Let W be a space. Then it is standard that $X \sqcap Y$ carries the initial topology relative to the maps p^*q , and q^*p . The typical map

$$W \longrightarrow X \sqcap Y$$

will be denoted by $\langle h, k \rangle$, where $h \in M(W, X)$ and $k \in M(W, Y)$ with $ph = qk$, thus $\langle h, k \rangle(w) = (h(w), k(w))$ where $w \in W$.

The k -ification of $X \sqcap Y$ will be denoted by $X \sqcap_k Y$. It follows from Theorem 2 that $X \sqcap_k Y$ carries the k -sense initial topologies relative to $k(p^*q)$, and $k(q^*p)$.

1.3.1. Exponential Rules for k -spaces. If X and Y are spaces, $M(X, Y)$ will denote the set of all maps from X to Y . In cases where it is a topological space, it should be assumed to have the compact-open topology.

Lemma 1. If X is a k -space, and C is compact Hausdorff, then $X \times C$ is a k -space.

Proof. We need to prove that the identity function

$$1 : X \times C \longrightarrow X \times_k C$$

is continuous. The first step is to show that $X \times C$ has the final topology relative to all maps

$$h \times 1_C : K \times C \longrightarrow X \times C$$

where K is compact Hausdorff, and $h \in M(K, X)$.

Let Z be an arbitrary space and

$$f : X \times C \longrightarrow Z$$

be a function. We will assume that

$$f \circ (h \times 1_C) : K \times C \longrightarrow Z$$

is continuous for every compact Hausdorff spaces K , and all h in $M(K, X)$. It follows by the proper condition for the category of all topological spaces (see [9, Ch. V. Lem. 3.1]). Then there is an associated map

$$u : K \longrightarrow M(C, Z)$$

determined by the rule

$$\begin{aligned} u(y)(c) &= f \circ (h \times 1_C)(y, c) \\ &= f(h(y), c) \\ &= (gh(y))(c) \end{aligned}$$

where $y \in K$ and the function

$$g : X \longrightarrow M(C, Z)$$

corresponds to f by the rule $g(x)(c) = f(x, c)$, $x \in K$ and $c \in C$. Hence $g \circ h = u$ is continuous for all choices of K and h . The Universal Property, associated with the k -space topology on X , implies that

$$g : X \longrightarrow M(C, Z)$$

is continuous.

The admissible condition for the category of all spaces [Ch.V. Cor.3.5 [9]] now ensures that f is continuous. Hence the maps

$$h \times 1_C : K \times C \longrightarrow X \times C$$

satisfy the Universal Property associated with the required final topology on $X \times C$, so $X \times C$ has that topology.

We will again assume that K is a compact Hausdorff space and that $h : K \longrightarrow X$ is a map. Then

$$h \times 1_C : K \times C \longrightarrow X \times C$$

and

$$h \times 1_C : k(K \times C) \longrightarrow k(X \times C) = X \times C$$

are continuous. Now $K \times C$ is compact Hausdorff, so it is a k -space, i.e. $k(K \times C) = K \times C$; hence

$$h \times 1_C : k(K \times C) \longrightarrow X \times_k C$$

is continuous.

Now this last map is the composite

$$\begin{array}{ccc} K \times C & \xrightarrow{h \times 1_C} & X \times C \\ & \searrow 1 \circ (h \times 1_C) & \downarrow 1 \\ & & X \times_k C \end{array}$$

so it follows by the Universal property established earlier in this proof, that

$$1 : X \times C \longrightarrow X \times_k C$$

is continuous.

Hence $X \times C = X \times_k C$, and so is k -space. \square

Theorem 3. *Let X, Y and Z be k -spaces. Then*

$$f : X \times_k Y \longrightarrow Z$$

is a continuous function if and only if

$$g : X \longrightarrow kM(Y, Z)$$

is continuous, where $f(x, y) = g(x)(y)$ for all $x \in X, y \in Y$, and $M(Y, Z)$ is the space of continuous functions from Y to Z with the compact open topology.

Proof. The proof follows immediately from the next three results. \square

1.3.2. **Proper Condition for k -spaces.**

Proposition 8. The Proper Condition *Let X, Y and Z be k -spaces, and $f : X \times_k Y \longrightarrow Z$ be continuous. Then the rule $g(x)(y) = f(x, y)$, where $x \in X$ and $y \in Y$, determines a well defined and continuous function*

$$g : X \longrightarrow kM(Y, Z).$$

Proof. Fixing $x \in X$, let

$$g(x) : Y \longrightarrow Z$$

be defined by $g(x)(y) = f(x, y)$ where $y \in Y$. Then $g(x)$ is clearly a well defined function. Now we need to prove that $g(x)$ is continuous. If $c_x : X \longrightarrow Y$ is the constant map at value x , then

$$\langle c_x, 1_Y \rangle : Y \longrightarrow X \times_k Y$$

defined by $\langle c_x, 1_Y \rangle(y) = (x, y)$ is a continuous function (see Remark 2). It follows that $g(x) = f \circ \langle c_x, 1_Y \rangle$ is continuous.

Let C be compact Hausdorff space, and $\alpha \in M(C, X)$. We wish to prove that $g \circ \alpha$ is continuous for all choices of α . For then, by the Universal Property associated with the k -topology on X , g is continuous.

Thus

$$\alpha \times 1_Y : C \times Y \longrightarrow X \times Y$$

is continuous, and $k(\alpha \times 1_Y)$ is continuous. So that

$$f \circ k(\alpha \times 1_Y) : C \times Y \longrightarrow Z$$

is continuous by previous Lemma. It follows by the proper condition in the ordinary sense; see [9, Lem. 3.1, Pg. 158], that

$$h : C \longrightarrow M(Y, Z)$$

is continuous, where

$$h(c)(y) = f(\alpha(c), y) = g(\alpha(c))(y),$$

where $c \in C$ and $y \in Y$. Hence

$$h(c) = g(\alpha(c)) = (g \circ \alpha)(c).$$

So $g \circ \alpha = h$ is continuous for all $\alpha \in M(C, X)$, and the result follows, as explained earlier. \square

Proposition 9. *If Y and Z are k -spaces, then*

$$e : kM(Y, Z) \times_k Y \longrightarrow Z$$

is continuous.

Proof. Given that C is compact Hausdorff, and

$$\alpha : C \longrightarrow kM(Y, Z) \times_k Y$$

is continuous. We want to prove that $e \circ \alpha$ is continuous, where

$$\alpha(c) = (\alpha_1(c), \alpha_2(c)),$$

$\alpha_1 : C \longrightarrow kM(Y, Z)$, and $\alpha_2 : C \longrightarrow Y$ are continuous. Further, it follows by Proposition 4 that $\alpha_1 : C \longrightarrow M(Y, Z)$ is also continuous, and

$$\alpha_2^* : M(Y, Z) \longrightarrow M(C, Z),$$

$\alpha^*(h) = h \circ \alpha_2$ is continuous, where $h \in M(Y, Z)$. Now

$$e_C : M(C, Z) \times C \longrightarrow Z$$

is continuous since C is compact Hausdorff see [Ch.V, Lem. 3.9 [9]]. Then $e \circ \alpha$ is continuous because $e \circ \alpha = e \circ \langle \alpha_1, \alpha_2 \rangle = e_C \circ \langle \alpha_2^* \circ \alpha_1, 1_C \rangle$. Hence e is continuous. \square

1.3.3. *Admissible Condition.*

Proposition 10. The Admissible Condition *If X , Y and Z are k -spaces, and $g : X \longrightarrow kM(Y, Z)$ is continuous, then $f : X \times_k Y \longrightarrow Z$ is continuous where $f(x, y) = g(x)(y)$.*

Proof. The proof follows because f is the composite

$$X \times_k Y \xrightarrow{g \times_k 1_Y} kM(Y, Z) \times_k Y \xrightarrow{e} Z,$$

in which

$$\begin{aligned} e(g \times_k 1_Y)(x, y) &= e(g(x), 1_Y(y)) \\ &= e(g(x), y) \\ &= g(x)(y) \\ &= f(x, y). \end{aligned}$$

Hence f is continuous. \square

Theorem 4 (Ch.V, Th.3.9 [9]). *Let X and Y be Hausdorff spaces and Z an arbitrary space. If either of the following conditions are satisfied:*

- (a) Y is locally compact or
- (b) X and Y satisfy the first axiom of countability,

then

$$f : X \times Y \longrightarrow Z$$

is a continuous function if and only if

$$g : X \longrightarrow M(Y, Z)$$

is continuous, where $f(x, y) = g(x)(y)$ for all $x \in X$, $y \in Y$, and $M(Y, Z)$ is the space of continuous functions from Y to Z with the compact open topology.

We notice that the inconvenient assumptions built into the admissible condition of Theorem 4 avoided in the corresponding result here i.e. Theorem 3.

Theorem 5. *If*

$$q : Y \longrightarrow B$$

is a Hurewicz fibration in the sense of usual category of spaces, then

$$kq : kY \longrightarrow kB$$

is a Hurewicz fibration in the k – sense.

Proof. Let A be a k – space, and

$$f : A \times \{0\} \longrightarrow kY,$$

and

$$F : A \times I \longrightarrow kB$$

be maps such that $F(a, 0) = kq(f(a, 0))$ for all $a \in A$. We wish to prove that there is a map

$$F^\sim : A \times I \longrightarrow kY$$

such that $F^\sim(a, 0) = f(a, 0)$ for $a \in A$, and $kq \circ F^\sim = F$.

Taking 1_Y , and 1_B to be the identity functions $kY \longrightarrow Y$, and $kB \longrightarrow B$, respectively, then

$$1_Y \circ f : A \times \{0\} \longrightarrow Y,$$

and

$$1_B \circ F : A \times I \longrightarrow B$$

are maps such that $1_B \circ F(a, 0) = q \circ (1_Y \circ f)(a, 0)$, for all $a \in A$. Then it follows from the covering homotopy property for p that we can find a map

$$H : A \times I \longrightarrow Y$$

such that $1_B \circ F = p \circ H$ and $H(a, 0) = 1_Y f(a, 0)$, for all $a \in A$. We define

$$F^\sim : A \times I \longrightarrow kY$$

as having the same underlying function as H . Now $A \times I$ is a k – space by Lemma 1, so H is continuous by Proposition 3. Hence the result follows. \square

2. MAPPING SPACES AND FIBREWISE HOMOTOPY THEORY

Definition 1. A topological space B is said to be weak Hausdorff if

$$\Delta_B = \{(b, b) \mid b \in B\} \subset B \times B,$$

is closed in $B \times_k B$.

Definition 2. If Z is a space, we will define Z^\sim as the set $Z \cup \{\omega\}$ where $\omega \notin Z$. We give Z^\sim the topology whose closed sets are Z^\sim itself, and the closed sets of Z . Let C be a closed subspace of Y , and

$$f : C \longrightarrow Z$$

be a map, so f is a partial map from Y to Z . Then there is an associated map

$$f^\sim : Y \longrightarrow Z^\sim$$

defined by the rule

$$f^\sim(y) = \begin{cases} f(y) & \text{if } y \in C \\ \omega & \text{otherwise.} \end{cases}$$

Remark 4. Let B be a T_1 -space, and $q : Y \longrightarrow B$ be a map. We define the set

$$Y!Z = \bigcup_{b \in B} M(Y \mid b, Z)$$

where $q^{-1}(b) = Y \mid b$, and the function

$$q!Z : Y!Z \longrightarrow B$$

is the projection that sends all maps

$$Y \mid b \longrightarrow Z$$

to b , for all $b \in B$. We know that B is a T_1 -space, so each fibre $Y \mid b$ is closed in Y . It follows that if $f \in M(Y \mid b, Z)$, then $i(f) = f^\sim$ defines a function

$$i : Y!Z \longrightarrow M(Y, Z^\sim).$$

We define the modified compact-open topology on $Y!Z$ as being the initial topology relative to i , and $q!Z$. Thus we define the free range mapping space $Y!Z$ as having a subbase consisting of all sets of the form $(q!Z)^{-1}(U)$, where U is open in B , and all sets of the form

$$W(A, V) = \{f \in Y!Z \mid f(A \cap \text{dom}(f)) \subset V\},$$

where A ranges over the compact subsets of Y , and V ranges over the open subsets of Z .

We now introduce a k -version of the free range mapping space $Y!Z$, i.e. $k(Y!Z)$. Thus this space carries the initial topology relative to $k(q!Z)$, and $k(i)$ in the sense of k -spaces, i.e. the k -ification of the previously defined topology on $Y!Z$.

Remark 5. *For the rest of this chapter all spaces used should be assumed to be k -spaces, and all constructions and definitions should be understood in that sense. Thus $X \times Y$, $X \sqcap Y$, $M(X, Y)$, $X!Y$ and $p!Y$ will refer to concepts that were previously denoted by $X \times_k Y$, $X \sqcap_k Y$, $kM(X, Y)$, $k(X!Y)$ and $k(p!Y)$, respectively. The term *Hurewicz fibration* refers to a map between k -spaces that has the covering homotopy property relative to incoming maps from k -spaces.*

2.1. Fibred Exponential Law for k -spaces.

Theorem 6. Fibred Exponential Law for k -spaces *Let X , Y , Z and B be k -spaces, with B weak Hausdorff, and $p : X \rightarrow B$, $q : Y \rightarrow B$, and $r : Z \rightarrow B$ be maps. Then there is a bijective correspondence between*

(a) *maps*

$$f^> : X \sqcap Y \rightarrow Z,$$

and

(b) *fibrewise maps*

$$f^< : X \rightarrow Y!Z$$

determined by the rule $f^>(x, y) = f^<(x)(y)$ where $p(x) = q(y)$.

Proof. There is a map

$$p \times q : X \times Y \rightarrow B \times B,$$

and the weak Hausdorff condition ensures that Δ_B is closed in $B \times B$. Hence the underlying set of $X \sqcap Y$, i.e.

$$(p \times q)^{-1}(\Delta_B),$$

is a closed subspace of $X \times Y$, so it follows that our theory of partial maps from Y to Z , with closed domains, is relevant to the situation under consideration.

Let

$$f^> : X \sqcap Y \rightarrow Z$$

be a map. Then $f^>$ determines a map

$$g^> = (f^>)^\sim : X \times Y \rightarrow Z^\sim$$

by the rule $g^>(x, y) = f^>(x, y)$, where $p(x) = q(y)$, and $g^>(x, y) = w$ otherwise. We know by the proper condition (Proposition 8) that there is an associated map

$$g^< : X \rightarrow M(Y, Z^\sim)$$

defined by $g^<(x)(y) = g^>(x, y)$, where $x \in X$, and $y \in Y$. So

$$g^<(x)(y) = w$$

if and only if

$$p(x) \neq q(y).$$

We now define

$$f^< : X \rightarrow Y!Z$$

by $f^<(x)(y) = g^<(x)(y)$ only in the case where $p(x) = q(y)$. Then

$$f^<(x)(y) = g^<(x)(y) = g^>(x, y) = f^>(x, y).$$

However, $f^<(x)(y)$ is undefined when $p(x) \neq q(y)$. If $p(x) = b$, then $f^<(x)(y)$ is defined for all $y \in Y \mid b$. i.e. $f^<(x) \in Y!Z$, and $(q!Z)(f^<(x)) = b$. So $(q!Z) \circ f^< = p$, and $(q!Z) \circ f^<$ is continuous. Also, recalling our definition of the topology on $Y!Z$, $i \circ f^< = g^<$ is continuous. It follows by the Universal Property of the k -sense initial topology on $Y!Z$, and Proposition 3, that $f^<$ is continuous. The argument is reversible, and so the proof is complete. \square

Remark 6. If X and Y are spaces, then $[X, Y]$ will denote the set of homotopy classes of free maps from X to Y . If X and Y are based spaces, then $M^o(X, Y)$ will denote the set of based maps from X to Y , with the of course k -ified compact-open topology, and $[X, Y]^o$ will be denote the set of based homotopy classes. If Y and B are based spaces, and $q : Y \longrightarrow B$ is a map, the set of based sections to q , i.e.

$$Sec^o(q) = \{f \in M^o(B, Y) \mid q \circ f = 1_B\}$$

will be equipped with the of course k -ified compact-open topology. In addition, if B and Z have basepoints $b_o \in B$ and $z_o \in Z$, then the constant map

$$c_{z_o} : Y \mid b_o \longrightarrow Z$$

is defined by $c_{z_o}(y) = z_o$. We take c_{z_o} as basepoint for $Y!Z$. The space $M(X, A; Y, B)$ will denote the set of maps from X to Y for which $f(A) \subseteq B$, again with the k -ified compact-open topology, and $[X, A; Y, B]$ for the corresponding set of homotopy classes.

2.2. Vertical Homotopies and Sections.

Definition 3. Vertical Homotopy Let $q : Y \longrightarrow B$ be a map, and ℓ_o , and ℓ_1 be sections to q . A homotopy $F : B \times I \longrightarrow Y$ such that $F_t = F(-, t) : B \longrightarrow Y$ is a section to q , for all $t \in I$, will be said to be a vertical homotopy. The sections ℓ_o , and ℓ_1 will be said to be vertically homotopic if there is a vertical homotopy from ℓ_o to ℓ_1 . I will write $\Pi_o(Sec^o(q))$ for the corresponding set of homotopy classes.

Corollary 5. Section Rule. Let (Z, z_o) , and (B, b_o) be based spaces, B being weak Hausdorff, and $q : Y \longrightarrow B$ be a map.

- (a) If $\ell : (Y, Y \mid b_o) \longrightarrow (Z, z_o)$ is a map, then the rule $\ell^\bullet(b) = \ell(Y \mid b) \longrightarrow Z$, where $b \in B$, defines a based section ℓ^\bullet to $q!Z$. Thus $\ell^\bullet(b)(y) = \ell(y)$, where $q(y) = b$. Then there is a bijective correspondence

$$\theta : M(Y, Y \mid b_o; Z, z_o) \longrightarrow Sec^o(q!Z),$$

where $\theta(\ell) = \ell^\bullet$, $\ell \in M(Y, Y \mid b_o; Z, z_o)$.

(b) If

$$\ell_o, \ell_1 \in M(Y, Y|b_o; Z, z_o)$$

then $\ell_o \simeq \ell_1$ via homotopy

$$F : (Y \times I; (Y|b_o) \times I) \longrightarrow (Z, z_o),$$

if and only if $\ell_o^\bullet \simeq \ell_1^\bullet$ via a based vertical homotopy.

(c) The rule $[\ell] \rightsquigarrow [\ell^\bullet]$ defines a bijection

$$\lambda : [Y, Y|b_o; Z, z_o] \longrightarrow \Pi_o(\text{Sec}^o(q!Z)).$$

Proof.

(a) The domain of $\ell^\bullet(b)$ is $Y|b$ so $q \circ \ell = 1_B$. Also $\ell^\bullet(b_o) = \ell|(Y|b_o) = c_{z_o}$, so ℓ^\bullet is base point preserving. If $B \sqcap Y$ is defined as the pullback of 1_B , and

$$q : Y \longrightarrow B,$$

then the projection

$$\pi : B \sqcap Y \longrightarrow Y$$

is a homeomorphism. So we have a bijective correspondence between maps

$$\ell : Y \longrightarrow Z$$

maps

$$\ell \circ \pi : B \sqcap Y \longrightarrow Z$$

and, by the Fibred Exponential Law,

$$\ell^\bullet : B \longrightarrow Y!Z.$$

We notice that $\ell^\bullet(b)(y) = \ell \circ \pi(b, y) = \ell(y)$, where $q(y) = b$.

(b) It follows by arguments similar to those the proof of (a) that

$$F : (Y \times I; (Y|b_o) \times I) \longrightarrow (Z, z_o)$$

is continuous, if and only if

$$G : (B \times I, \{b_o\} \times I) \longrightarrow (Y!Z, c_{z_o})$$

is continuous, where $F(y, t) = G(b, t)(y)$, for all $y \in Y$, $t \in I$ and $b = q(y)$. Moreover, $F(Y|b_o \times I) = z_o$ if and only if $G(b_o \times I)(y) = z_o$ for all $y \in Y|b_o$ and

$$(B \sqcap Y) \times I \cong (B \times I) \sqcap Y,$$

thus $\ell_o^\bullet \simeq \ell_1^\bullet$ as required.

(c) This follows easily from (a) and (b). □

Remark 7. As an example, let $q : Y \longrightarrow B$ be a map, Z be a space where $z_o \in Z$. Then there is a function

$$\sigma_{z_o} : B \longrightarrow Y!Z,$$

where $\sigma_{z_o}(b)$ is the constant map from $Y|b \longrightarrow Z$ with value z_o .

Now σ_{z_o} corresponds, via Corollary 5, part(a), to the constant map $Y \longrightarrow Z$ value z_o . Hence σ_{z_o} is continuous. It is easily seen that it is also a section to $q!Z$.

We now introduce some fibrewise terminology. Fibrewise spaces in the free sense are simply maps of spaces into B . Let $p : X \rightarrow B$ and $q : Y \rightarrow B$ be fibrewise spaces in the free sense. Then a fibrewise map from $p : X \rightarrow B$ to $q : Y \rightarrow B$ in the free sense is, of course, a map $f : X \rightarrow Y$ such that $q \circ f = p$.

A fibrewise space in the based sense is a pair (p, s) , where $p : X \rightarrow B$ is a map, and $s : B \rightarrow X$ is a section to p . The reader can observe that if B is a point $*$, then $s : * \rightarrow X$ is essentially just the point $s(*) \in X$, so $(p : X \rightarrow *, s : * \rightarrow X)$ is essentially just the based space $(X, s(*))$.

If (p, s) and (q, t) are fibrewise based spaces, then $(p \sqcap q, \langle s, t \rangle)$ is also a fibrewise based space.

A fibrewise map in the based sense, from (p, s) to (q, t) is a map $f : X \rightarrow Y$ such that $q \circ f = p$ and $f \circ s = t$. The set of based maps of this sort will be denoted by $M_B(X, Y)$.

Definition 4. If $f, g \in M_B(X, Y)$ then a fibrewise based homotopy from f to g is a fibrewise map $F : X \times I \rightarrow Y$, and based homotopy such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$.

Thus a fibrewise based homotopy from f to g is just a homotopy in the ordinary sense from f to g , which is a fibrewise based map at each stage of the deformation. We then write $f \simeq_B g$.

Definition 5. The fibrewise tertiary system (q, s, m) consists of a fibrewise based space Y over B , i.e. a map $q : Y \rightarrow B$, and a section $s : B \rightarrow Y$ to q , and a fibrewise based map $m : Y \sqcap Y \rightarrow Y$, called a fibrewise multiplication.

Definition 6. The fibrewise multiplication m is fibrewise homotopy commutative if $m \simeq_B m \circ \tau$, where τ is the switching fibrewise homeomorphism $\tau : Y \sqcap Y \rightarrow Y \sqcap Y$ defined by $\tau(y, y') = (y', y)$, for $(y, y') \in Y \sqcap Y$.

Definition 7. The fibrewise multiplication m is fibrewise homotopy associative if

$$m(m \sqcap 1_Y) \simeq_B m(1_Y \sqcap m)$$

where

$$Y \sqcap Y \sqcap Y \xrightarrow{1_B \sqcap m} Y \sqcap Y \xrightarrow{m} Y,$$

and

$$Y \sqcap Y \sqcap Y \xrightarrow{m \sqcap 1_B} Y \sqcap Y \xrightarrow{m} Y.$$

Definition 8. The fibrewise multiplication m has a fibrewise homotopy identity, or satisfies the Hopf condition if

$$m(1_Y \sqcap (s \circ q)) \Delta \simeq_B 1_Y \simeq_B m((s \circ q) \sqcap 1_Y) \Delta,$$

where $\Delta : Y \rightarrow Y \sqcap Y$ denotes the diagonal map,

$$Y \xrightarrow{\Delta} Y \sqcap Y \xrightarrow{1_Y \sqcap (s \circ q)} Y \sqcap Y \xrightarrow{m} Y,$$

and

$$Y \xrightarrow{\Delta} Y \sqcap Y \xrightarrow{(s \circ q) \sqcap 1_Y} Y \sqcap Y \xrightarrow{m} Y.$$

Definition 9. *The fibrewise based map $\mu : Y \longrightarrow Y$ is a fibrewise homotopy inversion for the fibrewise multiplication m if*

$$m(1_Y \sqcap \mu)\Delta \simeq_B s \circ q \simeq_B m(\mu \sqcap 1_Y)\Delta,$$

where

$$Y \xrightarrow{\Delta} Y \sqcap Y \xrightarrow{1_Y \sqcap \mu} Y \sqcap Y \xrightarrow{m} Y,$$

and

$$Y \xrightarrow{\Delta} Y \sqcap Y \xrightarrow{\mu \sqcap 1_Y} Y \sqcap Y \xrightarrow{m} Y.$$

A homotopy associative fibrewise tertiary system satisfying the Hopf condition, and for which the fibrewise multiplication admits an inversion, is called a fibrewise H -group. If a fibrewise H -group is fibrewise homotopy commutative, then it will be said to be homotopy Abelian.

More details concerning fibrewise homotopy are given in a locus classicus [3], [4] [10], [11], [13].

Proposition 11. *Let Z be an H -group, B a weak Hausdorff space, and $q : Y \longrightarrow B$ a map. Then there is a fibrewise map*

$$n : Y!Z \times Y!Z \longrightarrow Y!Z,$$

$n(f, g) = m(f \times g)\Delta_b$, where $b \in B$, $f, g \in M(Y|b, Z)$, m denotes the operation on Z , Δ_b is the diagonal map for $Y|b$, and $m(f \times g)\Delta_b$ is the following composite of maps

$$Y|b \xrightarrow{\Delta_b} Y|b \times Y|b \xrightarrow{f \times g} Z \times Z \xrightarrow{m} Z.$$

Then, defining σ_e as in the example of Remark 7, the tertiary system

$$(q!Z, \sigma_{(e)}, n)$$

is a fibrewise H -group. Further, if Z is homotopy Abelian, then $(q!Z, \sigma_e, n)$ is fibrewise homotopy Abelian.

Proof. If Y is a space and Z is an H -group, then the operation

$$n : M(Y, Z) \times M(Y, Z) \longrightarrow M(Y, Z),$$

defined by $n(f, g) = n \circ (f \times g) \circ \Delta_Y$, together with the identity map $c_e : Y \longrightarrow Z$, makes $M(Y, Z)$ into an H -group in an obvious fashion. If Z is homotopy Abelian, then so also is $M(Y, Z)$. The proof of this proposition is a direct generalization of that argument, using the fibred exponential law of Theorem 5, rather than the usual exponential law for spaces. \square

Proposition 12. *If $(q : Y \longrightarrow B, t : B \longrightarrow Y, m : Y \sqcap Y \longrightarrow Y)$ is a fibrewise homotopy Abelian H -group, then $\text{Sec}^o(q)$ is a homotopy Abelian H -group. Thus if $t_1, t_2 \in \text{Sec}^o(p)$, the operation $+_B$ on $\text{Sec}^o(q)$ is defined by*

$$t_1 +_B t_2 = m \circ \langle t_1, t_2 \rangle,$$

and the identity point for $\text{Sec}^o(q)$ is t .

Proof. The proof is routine. \square

Corollary 6. *If (Z, z_o) is an Abelian H – group, (B, b_o) is based weak Hausdorff space, and $q : Y \longrightarrow B$ is a map, then $\text{Sec}^o(q!Z)$ is an Abelian H – group, and $\Pi_o(\text{Sec}^o(q!Z))$ is an Abelian group.*

Theorem 7. *If (Z, z_o) is an Abelian H – group, (B, b_o) is based weak Hausdorff space, and $q : Y \longrightarrow B$ is a map, then*

(a) *the set*

$$[Y, Y|b_o; Z, z_o]$$

carries an Abelian group structure, which is defined by pointwise addition of homotopy classes, and

(b) *the bijection of corollary 5, part (c), i.e.*

$$\lambda : [Y, Y|b_o; Z, z_o]^o \approx \Pi_o(\text{Sec}^o(q!Z))$$

is an isomorphism.

Proof. (a) is routine. (b) The two group structures are both induced by the H -group structure on Z ; it is routine to verify that, as expected, λ is an isomorphism. \square

Proposition 13. *Let B , Y and Z be spaces and B be weak Hausdorff. If $q : Y \longrightarrow B$ is a Hurewicz fibration, then $q!Z$ is also a Hurewicz fibration.*

Proof. This is just the argument that proves Theorem 4.1 of [4], but reinterpreted in k – context. We assume that

$$F : A \times I \longrightarrow B$$

is a homotopy and the restriction $F|A \times 0$ is denoted by F_o . We then have pullback spaces $(A \times I) \sqcap Y$, and $(A \times 0) \sqcap Y$, induced by the homotopy F and the map F_o , respectively, and associate projections

$$\begin{aligned} F^*q : (A \times I) \sqcap Y &\longrightarrow A \times I, \\ (F_o)^*q : (A \times 0) \sqcap Y &\longrightarrow A \times I, \end{aligned}$$

$$\begin{aligned} q^*F : (A \times I) \sqcap Y &\longrightarrow Y, \\ q^*F_o : (A \times 0) \sqcap Y &\longrightarrow Y, \end{aligned}$$

and such that

$$q \circ (q^*F) = F \circ (F^*q)$$

and

$$q \circ (q^*F_o) = F_o \circ (F_o)^*q.$$

We recall that $(A \times 0) \sqcap Y$ is a retract of $(A \times I) \sqcap Y$. The proof of this, in the usual topological context, is given in [B6, Lm. 4.2]; the k – case proof is similar. Let

$$k^< : A \times 0 \longrightarrow Y!Z$$

be a map such that $(q!Z) \circ k^< = F_o$. It follows, by the Fibred Exponential Law for k – spaces, that there is an associated map

$$k^> : (A \times 0) \sqcap Y \longrightarrow Z$$

defined by $k^>(a, 0, y) = k^<(a, 0)(y)$ where $(a, 0, y) \in (A \times 0) \sqcap Y$. Now $(A \times 0) \sqcap Y$ is known to be a retract of $(A \times I) \sqcap Y$ (compare with [4, Lemma 4.2]).

Let

$$R : (A \times I) \sqcap Y \longrightarrow (A \times 0) \sqcap Y$$

be a retraction. Then the composite

$$k^> \circ R : (A \times I) \sqcap Y \longrightarrow Z$$

corresponds, via the fibred exponential law to

$$K : A \times I \longrightarrow Y!Z,$$

where $K(a, t)(y) = (k^> \circ R)(a, t, y)$, and $(a, t, y) \in (A \times I) \sqcap Y$. Then K is fibrewise over B , i.e. $(q!Z)K(a, t) = F(a, t)$, so $(q!Z) \circ K = F$. Also if $(a, 0, y) \in (A \times 0) \sqcap Y$,

$$K(a, 0)(y) = (k^> \circ R)(a, 0, y) = k^>(a, 0, y) = k^<(a, 0)(y).$$

So $K(a, 0) = k^<(a, 0)$ for $a \in A$. i.e. K extends $k^<$. Thus K lifts F and extends $k^<$, and $q!Z$ is a Hurewicz fibration. \square

3. APPLICATIONS TO HOMOTOPY THEORY

We will now compare the main result of [4] to the result of our section. **In this section we do not assume that spaces are k -spaces, unless we specifically say so.**

Theorem 8. Fibred Exponential Law. [4, Th. 3.3] *Let B be a Hausdorff space, Z be a space, and $p : X \longrightarrow B$ and $q : Y \longrightarrow B$ be a maps.*

- (a) **Proper Condition:** *If $f^> : X \sqcap Y \longrightarrow Z$ is a map, then the rule $f^<(x)(y) = f^>(x, y)$ determines a fibrewise map $f^> : X \longrightarrow Y!Z$, where $p(x) = q(y)$. Thus $f^<$ is a map such that $(q!Z) \circ f^< = p$.*
- (b) **Admissible Condition:** *Let us assume that either*
 - (i) *(X, Y) is an exponential pair of spaces, or*
 - (ii) *W is a space, $p : B \times W \longrightarrow B$ the projection, and $Y \times W$ a k -space. Then, given a fibrewise map $f^> : X \longrightarrow Y!Z$, the above rule determines a map $f^> : X \sqcap Y \longrightarrow Z$.*

We notice that the inconvenient assumptions built into the admissible condition of Theorem 8 avoided in the corresponding result here i.e. Theorem 6.

3.1. Section Rule.

Corollary 7. Section Rule. [4, Cor.3.4] *Let B be a Hausdorff space, and $q : Y \longrightarrow B$ be a map.*

- (a) *If $l : Y \longrightarrow Z$ is a map, then the rule $l^\bullet(b) = l|(Y|b) : Y|b \longrightarrow Z$, where $b \in B$, defines a section l^\bullet to $q!Z$. Equivalently, we may define l^\bullet by $l^\bullet(b)(y) = l(y)$, where $q(y) = b$.*

- (b) If Y is k -space and l^\bullet is a section to $q!Z$, then the rule stated in (a) determines a map $l : Y \rightarrow Z$.

The corresponding result in this work is Corollary 5.

Theorem 9. [4, Th. 8.1(b)] *There is a canonical bijection:*

$$\theta : H^m(Y, Y|b; G) \rightarrow \Pi_o(\text{Sec}^o(p!K(G, m)))$$

where the map θ is determined by the rule $\theta[l] = [l^\bullet]$, where $l^\bullet(b)(y) = l(y)$ and $q(y) = b$.

3.2. Ω -spectrum of Eilenberg-MacLane spaces. If we follow the Ω -spectrum of Eilenberg-MacLane spaces to cohomology, then the associated cohomology groups are defined by

$$H^m(Y, Y|b; G) = [Y, Y|b_o; K(G, m), e],$$

for more details concerning this spectra the reader can see [12, Def. 8.4.6].

Corollary 5 of this paper gives the k -version of both Corollary 7 and Theorem 8, and our Theorem 6 improves on Theorem 8.1(b) by showing that the bijection of that result is actually an isomorphism.

Hence we have established the k -version foundation of the application to the cohomology of fibrations that is discussed in Ch. 8 of [B6].

Theorem 10. [4, Th. 4.1]. *Let B, Y and Z be spaces, where B is Hausdorff and Y is locally compact Hausdorff. If $q : Y \rightarrow B$ is a Hurewicz fibration, then $q!Z$ is also a Hurewicz fibration.*

The reader can compare that result with the Proposition 12, and observe that Proposition 12 is free of the inconvenient assumption that Y is locally compact and Hausdorff.

We now consider the first group of applications given in [4], i.e. Theorems 10, 11 and 12.

Theorem 11. [4, Th. 5.1]. *Let A be a k -space and B be a Hausdorff space. If $q : Y \rightarrow B$ is an identification and $f : A \rightarrow B$ is a map, then $f^*p : Y \sqcap A \rightarrow A$ is an identification.*

Theorem 12. [4, Th. 6.1]. *Let $q : Y \rightarrow B$ be a Hurewicz fibration, where B is a Hausdorff space and Y is locally compact Hausdorff. If $A \rightarrow B$ is a closed cofibration, then $Y|A \rightarrow Y$ is also a closed cofibration.*

Remark 8. *Let $q : Y \rightarrow B$ be a map and $t : B \rightarrow Y$ be a section to q . If $f : A \rightarrow B$ is a map, then*

$$\sigma : A \rightarrow Y \sqcap A,$$

defined by $\sigma(a) = (tf(a), a)$, for all $a \in A$, is a section to the projection $f^*q : Y \sqcap A \rightarrow A$.

Theorem 13. [4, Th.7.1]. *Let $q : Y \longrightarrow B$ be a Hurewicz fibration, with closed cofibration section t , B be Hausdorff and Y locally compact Hausdorff. If $f : A \longrightarrow B$ is a map, then $f^*q : A \square A \longrightarrow A$ is a Hurewicz fibration with a closed cofibration section σ .*

Remark 9. *If we modify those proofs by assuming that all spaces are k – spaces and B is weak Hausdorff, and replacing the above results by our Theorems 5, we obtain the following analogous results.*

Theorem 14. *Let A be a k – space and B be a weak Hausdorff space. If $q : Y \longrightarrow B$ is an identification and $f : A \longrightarrow B$ is a map, then $f^*q : Y \square A \longrightarrow A$ is an identification.*

Theorem 15. *Let $q : Y \longrightarrow B$ be a Hurewicz fibration, where B is a weak Hausdorff space and Y is k – space. If $A \longrightarrow B$ is a closed cofibration, then $Y|A \longrightarrow Y$ is also a closed cofibration.*

Theorem 16. *Let $q : Y \longrightarrow B$ be a Hurewicz fibration, with closed cofibration section t , B be weak Hausdorff and Y k – space. If $f : A \longrightarrow B$ is a map, then $f^*q : A \square A \longrightarrow A$ is a Hurewicz fibration with a closed cofibration section σ .*

The reader will notice that the locally compact Hausdorff assumption of Theorem 12 and 13 have now been eliminated.

4. MOORE-POSTNIKOV SYSTEM

Let G and H be Abelian groups and m and n be integers with $1 < n < m$. Then

$$q_1 : PK(G, m+1) \longrightarrow K(G, m+1)$$

will denote the path fibration over the Eilenberg-MacLane space $K(G, m+1)$ and $K(H, n+1)$ (see [14], Pgs. 75 and 99). Let (B, b_o) be a space with a basepoint. Then a 3-stage Postnikov tower $\tau(k_1, k_2) = p_1 \circ p_2$, over B and with fibres $K(G, m)$ and $K(H, n)$, consists of principal fibrations

$$p_1 : E_1 \longrightarrow B$$

and

$$p_2 : E_2 \longrightarrow E_1$$

with fibres $K(G, m)$ and $K(H, n)$ respectively. So p_1 is induced from q_1 by first k -invariant

$$k_1 : B \longrightarrow K(G, m+1),$$

i.e. $p_1 = k_1^*q_1$, and p_2 is induced by q_2 by the second k -invariant

$$k_2 : B \longrightarrow K(H, n+1),$$

i.e. $p_2 = k_2^*q_2$. The maps k_1 and k_2 are based mappings, where the identities of $K(G, m+1)$ and $K(H, n+1)$ are their base points.

The fibre of p_1 over b_o is

$$\{b_o\} \times (PK(G, m)|k_1(b_o)) = \{b_o\} \times \Omega(K(G, m+1)) = \{b_o\} \times K(G, m),$$

where Ω indicate the corresponding loop space. The fibre of

$$\tau(k_1, k_2) = p_1 \circ p_2 : E_2 \longrightarrow B$$

is the subspace of E_2 obtained by pulling back over $k_2|_{\{b_o\}} \times (K(G, m))$, i.e. $\{b_o\} \times (K(G, m) \cap (K(H, n+1)))$. So the fibre of $\tau(k_1, k_2)$ is $K(G, m) \times K(H, n)$ if and only if

$$k_2|_{\{b_o\}} \times K(G, m) : \{b_o\} \times K(G, m) \longrightarrow K(H, n+1)$$

is the constant map

$$c_e : \{b_o\} \times K(G, m) \longrightarrow K(H, n+1)$$

with value the identity e of $K(H, n+1)$.

The problem considered in [4], and in [3], is the classification of 3-stage Postnikov towers, with fibre $K(G, m) \times K(H, n)$, up to fibrewise homotopy equivalence. The map

$$q : PK(G, m+1) \longrightarrow K(G, m+1)$$

and the space $K(H, n+1)$ enable us to define a free range mapping space $PK(G, m+1)!K(H, n+1)$. The classifying space M_∞ used in [3] and [4] is just a path component of that space, i.e. the component that contains the constant map

$$c_e : K(G, m) \longrightarrow K(H, n+1).$$

It is shown in [3, Th.7.5], that $[B, M_\infty]^o$ classifies our 3-stage Postnikov tower up to a strong form of fibrewise homotopy equivalence.

Let $\varepsilon(K(G, m) \times K(H, n))$ denote the group of homotopy classes of self-homotopy equivalence of $K(G, m) \times K(H, n)$. It is shown in [?] that an orbit set of $[B, M_\infty]^o$, under an action of $\varepsilon(K(G, m) \times K(H, n))$, classifies the fibre homotopy equivalence classes of 3-stage towers as discussed above.

Theorem (M6) of [3]. We consider 3-stage towers over B with fibre

$$K(G, m) \times K(H, n),$$

is proved on p.98 of [3] using the general form of the fibrewise exponential law for k -spaces see [3, Th. 5.2]. However, it is more natural to prove a free range exponential law for k -spaces as in our Theorem 6. That state and proof goes as follows:

4.1. Classification Theorem.

Theorem 17. *Let $\tau(k_1, k_2) = p_1 \circ p_2$ be a $(K(G, m) \times K(H, n))$ -tower. If $k_1 \in M^o(B, K(G, m+1))$ and $g \in M(E_1, \{b_o\} \times K(G, m+1); K(H, n+1), \{e\})$, then there is a bijective correspondence between:*

- (a) $K(G, m) \times K(H, n)$ -towers $\tau(k_1, k_2)$, and
- (b) maps $k \in M^o(B, M_\infty)$ determined by the rule $k(b)(l) = k_2(b, l)$, where $(b, l) \in B \cap PK(G, m+1) = E_1$, i.e. $k(b) = q_1(l)$.

Proof. let $\tau(k_1, k_2) = p_1 \circ p_2$ be a 3-stage Postnikov tower, over a path connected and weak Hausdorff space B , as described above. It follows by Theorem 6 that

$$k_2 : B \square PK(G, m+1) \longrightarrow K(H, n+1)$$

determines a map

$$k : B \longrightarrow PK(H, m+1)!K(H, n+1),$$

where $k(b)(l) = k_2(b, l)$ and $k_1(b) = q_1(l)$.

Conversely if k is given then k_2 is determined in this way.

Then $\tau(k_1, k_2)$ has fibre $K(G, m) \times K(H, n)$ if and only if

$$k_2|_{\{b_o\}} \times K(G, m) = c_e,$$

which is true if and only if $k(b_o)(l) = e$, i.e. if

$$k(b_o)(l) = c_e : K(G, m) \longrightarrow K(H, n+1).$$

Now B is path connected, so $k(B)$ is in the path component of

$$PK(G, m+1)!K(H, n+1)$$

that contains c_e , i.e. $k(B) \subset M_\infty$. Then the result follows. \square

5. OPEN PROBLEMS AND QUESTIONS

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